Optimal Control of Transmon Qubit Gates in the Presence of Decoherence

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Quantum Dynamics and Control

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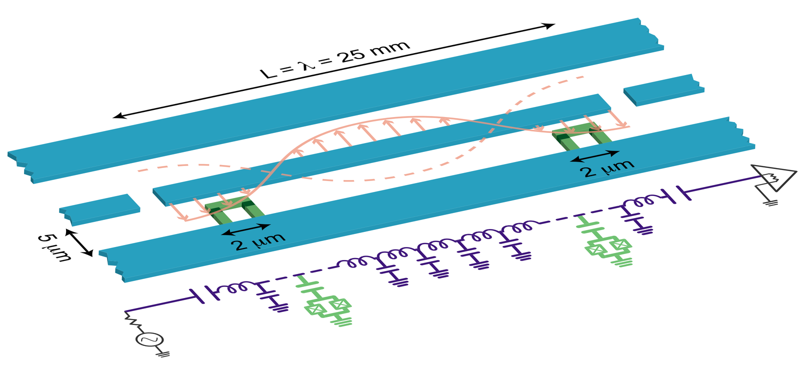
Abstract

We consider two transmon qubits [1] coupled via a cavity bus [2]. The strong coupling of each qubit to the shared cavity modes provides an indirect interaction that can be used to implement a two-quantum gate (e.g. CNOT, CPHASE). The analysis of such a system is often done with an effective spin-Hamiltonian. We show that generally, the approximation of limiting oneself to the two lowest-lying qubit levels and a limited number of cavity modes may not hold in all cases. Describing the system numerically allows us to take into account an arbitrary number of qubit and cavity excitations. Optimal control theory (OCT), specifically Krotov's method [3], is used to find microwave pulses that drive the full system in the desired way in the shortest possible amount of time. The complete system Hamiltonian allows for complex dynamics that OCT can fully exploit. We show results from such an optimization for a CPHASE and CNOT gate, for different pulse durations and central frequencies. Lastly, we also discuss decoherence, describing the system dynamics with a master equation in Lindblad form [4], and give an outlook on how OCT may find robust pathways.

① Two Transmon Qubits Coupled via Cavity Bus



superconducting qubits inside a transmission line resonator, Fig. from [5]

Parameters:

- $\omega_c = 8.3$ GHz
- $\omega_1 = 6.5$ GHz
- $\omega_2 = 6.6$ GHz
- $\alpha_1 = \alpha_2 = 150$ MHz
- $J = 5$ MHz
- $g_1 = g_2 = 100$ MHz
- $|\epsilon(t)| < 50$ MHz (if possible)

$$\hat{\mathbf{H}} = \underbrace{\omega_c \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}}}_{①} + \underbrace{\omega_1 \hat{\mathbf{b}}_1^\dagger \hat{\mathbf{b}}_1 + \omega_2 \hat{\mathbf{b}}_2^\dagger \hat{\mathbf{b}}_2 - (\alpha_1 \hat{\mathbf{b}}_1^\dagger \hat{\mathbf{b}}_1^\dagger \hat{\mathbf{b}}_1 \hat{\mathbf{b}}_1 + \alpha_2 \hat{\mathbf{b}}_2^\dagger \hat{\mathbf{b}}_2^\dagger \hat{\mathbf{b}}_2 \hat{\mathbf{b}}_2)}_{②} + \underbrace{J(\hat{\mathbf{b}}_1^\dagger \hat{\mathbf{b}}_2 + \hat{\mathbf{b}}_1 \hat{\mathbf{b}}_2^\dagger)}_{③} + \underbrace{g_1(\hat{\mathbf{b}}_1^\dagger \hat{\mathbf{a}} + \hat{\mathbf{b}}_1 \hat{\mathbf{a}}^\dagger) + g_2(\hat{\mathbf{b}}_2^\dagger \hat{\mathbf{a}} + \hat{\mathbf{b}}_2 \hat{\mathbf{a}}^\dagger)}_{④} + \underbrace{\epsilon^*(t)\hat{\mathbf{a}} + \epsilon(t)\hat{\mathbf{a}}^\dagger}_{⑤} \tag{1}$$

with ① the cavity harmonic oscillator, ② qubit anharmonic oscillators, ③ direct qubit-qubit coupling, ④ qubit-cavity coupling, and ⑤ cavity coupling to control field $\epsilon(t) \sim E_0 \cos(\omega_L t)$.

Note: Direct qubit-qubit coupling is weak; entanglement is primarily reached indirectly via interaction with the cavity ④.

② Optimization: Krotov Method

The Hamiltonian in Eq. (1) can be used to reach a large number of different two-qubit gates. We optimize for $\hat{\mathbf{O}}$ =CNOT, CPHASE by minimizing the functional J containing the gate fidelity F and a running cost ensuring monotonic convergence, with a scaling parameter λ_a and a shape function $S(t)$.

$$J[\{\phi_k\},\epsilon] = -F[\{\phi_k(T)\}] + \lambda_a \int_0^T \frac{\Delta\epsilon(t)}{S(t)}\,\mathrm{d}t\,, \qquad F = \frac{1}{16}\left|\sum_{k=1}^{4}\left\langle \phi_k \middle| \hat{\mathbf{O}}^\dagger \hat{\mathbf{U}} \middle| \phi_k \right\rangle\right|^2 \tag{2}$$

with $\Delta\epsilon = \epsilon^{\text{new}} - \epsilon^{\text{old}}$, for $|\phi_k\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, $|\phi_k(t)\rangle = \hat{\mathbf{U}}(t,0;\epsilon)\,|\phi_k\rangle$.
Pulse update formula [3]:

$$\Delta\epsilon(t) = \frac{S(t)}{\lambda_a}\mathfrak{Im}\left\{\sum_{k=1}^{4}\left\langle \chi_k^{\text{old}}(t) \middle| \frac{\partial \hat{\mathbf{H}}}{\partial \epsilon^{\text{new}}} \middle| \phi_k^{\text{new}}(t) \right\rangle + \underbrace{\sigma(t)\sum_{k=1}^{4}\left\langle \Delta\phi_k(t) \middle| \frac{\partial \hat{\mathbf{H}}}{\partial \epsilon^{\text{new}}} \middle| \phi_k^{\text{new}}(t) \right\rangle}_{\text{for choices of } F \text{ requiring second order}}\right\}, \tag{3}$$

$$\text{with}\quad |\chi_k(T)\rangle \equiv \frac{\partial F}{\partial \langle \phi_k(T)|} = \frac{1}{16}\sum_{k'=1}^{4}\left\langle \phi_{k'}^{\text{tgt}} \middle| \hat{\mathbf{U}} \middle| \phi_{k'} \right\rangle \left|\phi_k^{\text{tgt}}\right\rangle; \quad \left|\phi_k^{\text{tgt}}\right\rangle \equiv \hat{\mathbf{O}}\,|\phi_k\rangle\,.$$

③ Decoherence

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -i[\hat{\mathbf{H}},\hat{\rho}] + \kappa\mathcal{D}[\hat{\mathbf{a}}]\hat{\rho} + \sum_{i=1,2}\left[\sum_j \gamma_{i,j}\mathcal{D}[\hat{\mathbf{n}}^i_{j,j+1}]\hat{\rho} + 2\gamma_{\phi,i}\mathcal{D}[\hat{\mathbf{n}}^i_{\epsilon}]\hat{\rho} + \sum_j \gamma^{th}_{i,j}\mathcal{D}[\hat{\mathbf{n}}^i_{j+1,j}]\hat{\rho}\right] \tag{4}$$

$$\text{with}\quad \mathcal{D}[\hat{\mathbf{A}}]\hat{\rho} = \frac{1}{2}\left(2\hat{\mathbf{A}}\hat{\rho}\hat{\mathbf{A}}^\dagger - \hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}\hat{\rho} - \hat{\rho}\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}\right), \qquad \hat{\mathbf{n}}^i_{n,m} = |n\rangle\langle m|\,, \qquad \hat{\mathbf{n}}^i_{\epsilon} = \sum_n \epsilon_n |n\rangle\langle n|$$

The parameters $\gamma_{i,j}$, $\gamma_{\phi,i}$ and $\gamma^{th}_{j,i}$ are decay, dephasing and leakage rates, respectively. The cavity decay is described by κ.

It is straightforward to write the Krotov update equation (3) for Liouville space, using density matrices instead of states and using Eq. (4) for propagation. It can be shown that it is sufficient to use three density matrices as "basis states":

$$\rho^{(1)}_{ij} = \frac{2(N-i+1)}{N(N+1)}\delta_{ij}\,, \qquad \rho^{(2)}_{ij} = \frac{N^2-2}{N^2}\delta_{i1}\delta_{ji} + \frac{1}{N^2}\delta_{1j} + \frac{1}{N^2}\delta_{i1}\,, \qquad \rho^{(3)}_{ij} = \frac{1}{N}\delta_{ij}\,.$$

The most straightforward choice of a fidelity in this case is given by $F = \sum_{ikl}\left|\left(\rho^{(i)}(T) - \hat{\mathbf{O}}\rho^{(i)}\hat{\mathbf{O}}^\dagger\right)_{kl}\right|^2$. Instead of a fidelity based on the Frobenius norm, any other matrix norm may be used.

References

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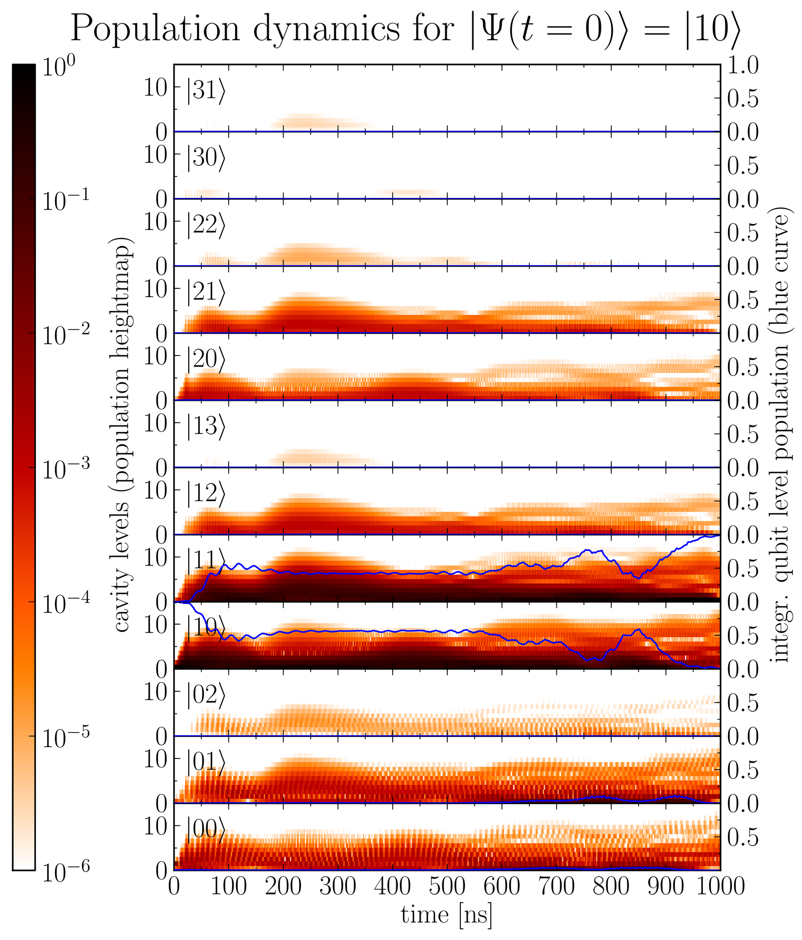
Overview

#	ω_L^{guess} [GHz]		T [ns]	E_0^{opt} [MHz]	Gate Error
CPHASE					
1	6.59	(qubit)	100	1000	$3.38 \cdot 10^{-1}$
2	6.59	(qubit)	250	700	$7.07 \cdot 10^{-2}$
3	8.30	(cavity)	250	600	$2.91 \cdot 10^{-2}$
4	6.59	(qubit)	1000	80	$2.41 \cdot 10^{-2}$ *
5	8.30	(cavity)	1000	50	$3.20 \cdot 10^{-2}$
6	8.30	(cavity)	5000	35	$9.70 \cdot 10^{-3}$ *
CNOT					
7	6.59	(qubit)	100	1100	$3.53 \cdot 10^{-1}$
8	6.59	(qubit)	250	850	$4.52 \cdot 10^{-2}$
9	8.30	(cavity)	250	700	$2.99 \cdot 10^{-2}$
10	6.59	(qubit)	1000	120	$1.03 \cdot 10^{-2}$ *
11	8.30	(cavity)	1000	120	$1.50 \cdot 10^{-3}$ *

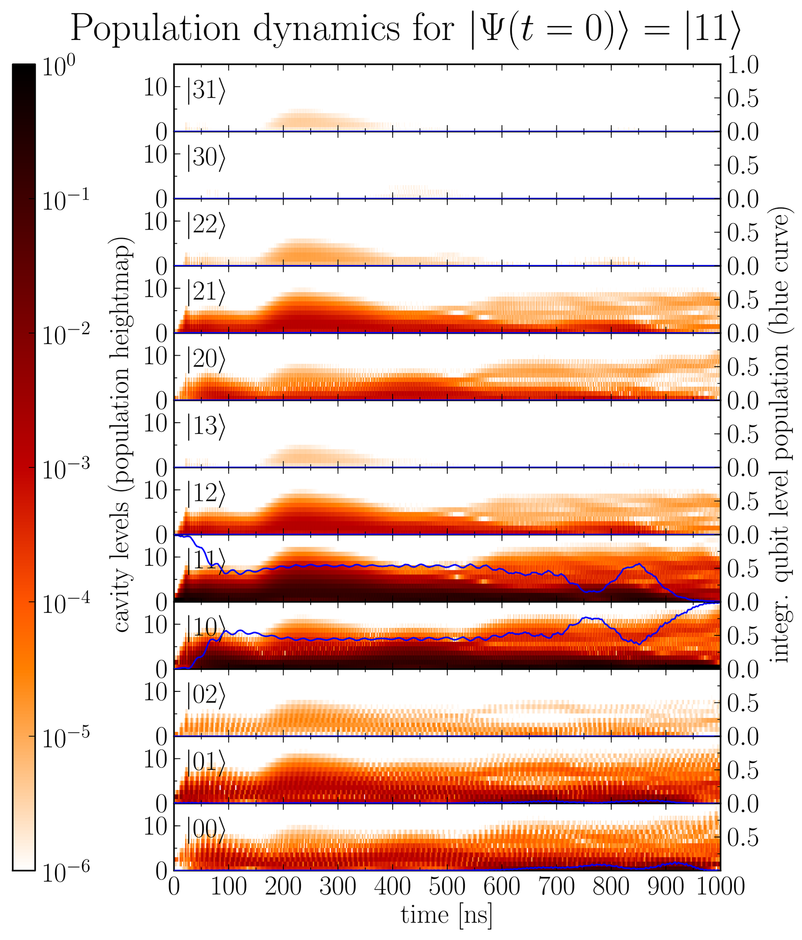
* Gate Error $< 1 \cdot 10^{-3}$ expected with further optimization

CNOT (pulse 10; $T = 1000$ ns, $F = 0.990$)

Population dynamics for $|\Psi(t=0)\rangle = |10\rangle$

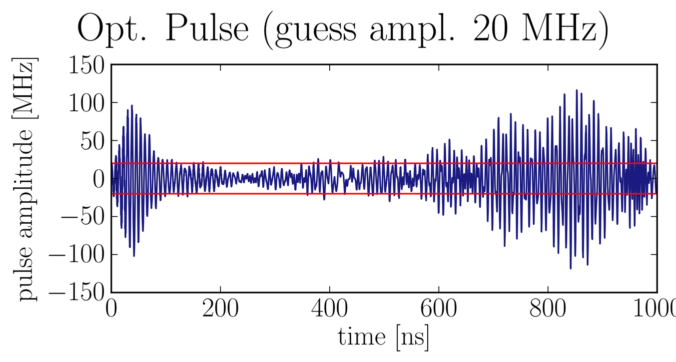


Population dynamics for $|\Psi(t=0)\rangle = |11\rangle$

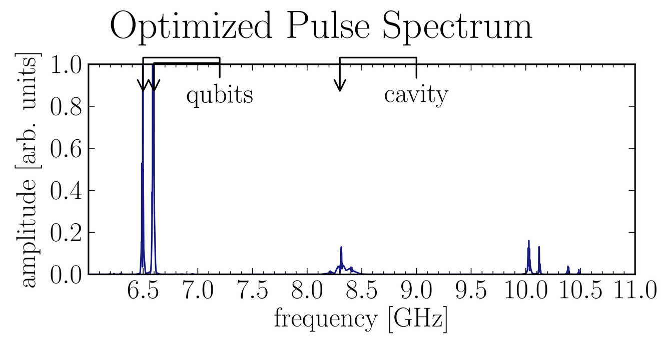


Note: integr. qubit level pop. for level $|ij\rangle$: $\sum_n |\langle\Psi(t)|\,ij\rangle \otimes |n\rangle|^2$

Opt. Pulse (guess ampl. 20 MHz)



Optimized Pulse Spectrum

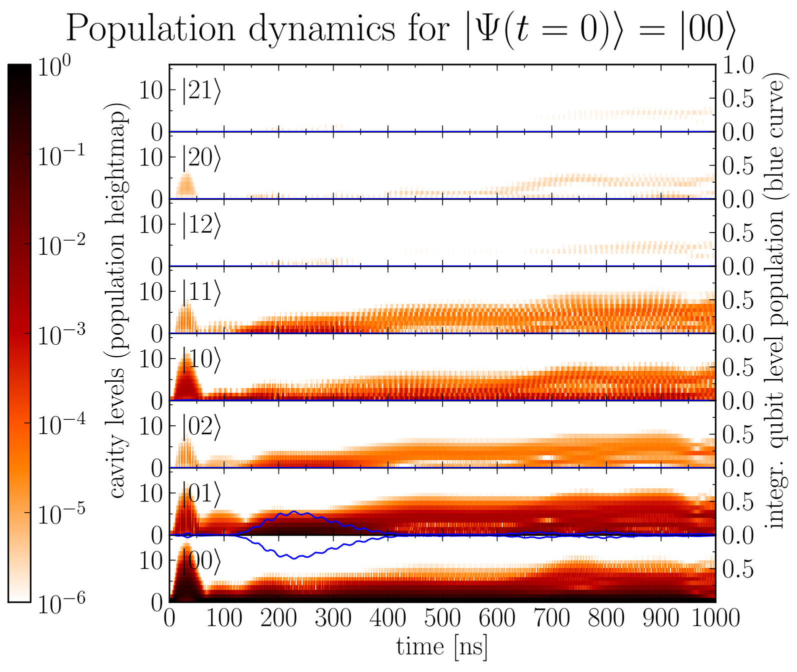


④ Optimization Results

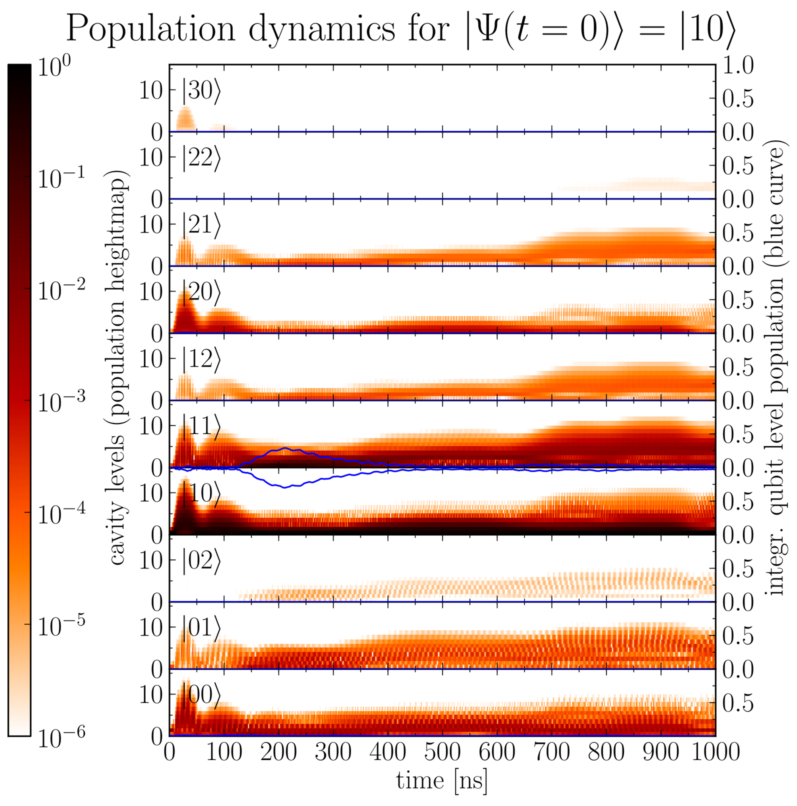
- Limits on fidelity: pulse duration too short to generate necessary entanglement (pulse 1,7); difficulty to restore cavity ground state if too many excitations (pulse 5).
- There is a tradeoff between pulse intensity and durations. $E_0 < 50$ MHz for $T > 1000$ ns
- Starting with different guess pulses, different mechanisms may be found implementing the same gate: pulse 5 implements CPHASE using only cavity-excitation (> 70 cavity levels), but almost no qubit excitation. Pulse 4 (shown below) implements the gate using qubit excitations, with minimal use of the cavity.
- Solutions may use 4-7 qubit levels, 15-100 cavity levels. Pulses primarily using the qubit frequency generally stay below 5 qubit levels and 25 cavity levels.

CPHASE (pulse 4; $T = 1000$ ns, $F = 0.976$)

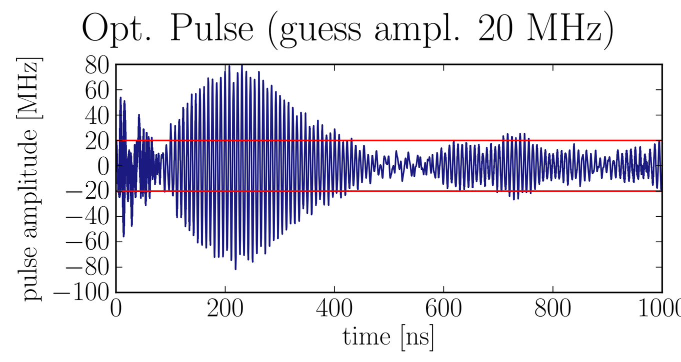
Population dynamics for $|\Psi(t=0)\rangle = |00\rangle$



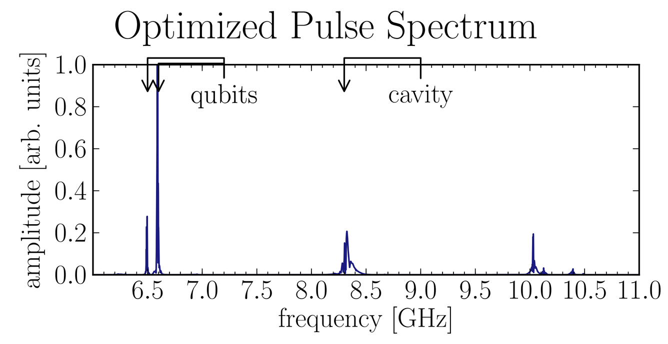
Population dynamics for $|\Psi(t=0)\rangle = |10\rangle$



Opt. Pulse (guess ampl. 20 MHz)



Optimized Pulse Spectrum



Yet unexplained observations on spectra:

- Optimized CNOT tends to excite qubits symmetrically, whereas CPHASE excites asymmetrically
- Pulses contain high frequency components. These become extremely pronounced on pulses obtained for either short pulse duration or from guess pulses centered around the cavity frequency.

⑤ Conclusions & Outlook

- The optimization of a CPHASE and CNOT gate for the transmon system illustrates the extremely rich dynamics that the Hamiltonian Eq. (1) provides. Different two-quantum gates can be implemented.
- Pulses may populate a significant number of higher qubit and cavity states, justifying use of the full Hamiltonian in lieu of an effective two-qubit description. Optimizing from appropriate guess pulses, the number of qubit and cavity levels can be kept reasonably low (e.g. 4 or 5 qubit levels, 20 cavity levels)
- Spectral components of optimized pulses must be better understood. Pulse peak amplitudes are still undesirably large. Finding pulses pulses with $\epsilon(t) < 50$ MHz may be achieved by adding a pulse intensity penalty to the functional Eq. (2)
- With the experience of the Hilbert space optimization, optimize in Liouville space, with decoherence. This can be done efficiently using the approach presented in section 3. The choice of fidelity may have a significant effect on the optimization success and should be explored systematically.
- Ultimately: Use the local invariants functional [6] in Liouville space to optimize for the two-qubit gate least susceptible to decoherence.